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ORIGINAL ARTICLE

A QUANTILE-BASED TEST FOR SYMMETRY OF WEAKLY DEPENDENT PROCESSES

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This article considers the problem of testing for symmetry of the marginal distribution of weakly dependent, stationary random processes. A quantile-based test for symmetry is proposed, which is easy to implement, requires no moment assumptions and has a standard asymptotic distribution. The finite-sample properties of the test are assessed by means of Monte Carlo experiments. An application to financial time series is also discussed.

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1. INTRODUCTION

The problem of testing for symmetry of a probability distribution about a specified or unspecified centre has been a topic of intensive research. This is not perhaps surprising in view of the fact that many nonparametric and robust statistical procedures rely to a certain extent on the assumption of symmetry. Symmetry is also important in terms of the definition and estimation of location since the centre of symmetry of a distribution is its only natural location parameter. In the context of model building, a check for symmetry is a useful diagnostic since asymmetry of the marginal distribution of the data would imply that certain types of parametric models (e.g. autoregressive moving average models with independent and symmetrically distributed innovations) would be statistically inadequate.

In the finance literature, symmetry is an implicit or explicit assumption in some widely used models, including, among others, the Sharpe–Lintner capital asset pricing model and the Black–Scholes option pricing model. With many empirical studies reporting significant evidence of asymmetry in the distributions of financial data, the adequacy of such models has been questioned, and extensions/modifications have been proposed to incorporate asymmetry in the models (see Peiró, 1999, for a useful discussion and many relevant references). Another well-known example from the economics literature relates to the question of whether real economic variables behave asymmetrically over the business cycle. Following Delong and Summers (1986), a substantial body of work has evolved in which different types of cyclical asymmetry are identified via the distributional asymmetry of relevant economic variables. In response to empirical findings of cyclical asymmetry, theoretical models have been developed in which asymmetry is generated endogenously (see, e.g. Acemoglu and Scott, 1997; Nieuwerburgha and Veldkamp, 2006). In addition to being a useful diagnostic, therefore, a test for symmetry may also be used as a means of evaluating the empirical validity of different economic hypotheses and models.

The vast majority of the work on tests for symmetry has focused on the case of i.i.d. data. However, a small number of studies have considered tests that are valid in the presence of weak dependence; relevant references

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include Chen *et al.* (2000), Bai and Ng (2001), Psaradakis (2003), Bai and Ng (2005), Delgado and Escanciano (2007) and Psaradakis (2008).

The present article contributes to this literature by considering a skewness-based test for symmetry (about an unknown centre) of the marginal distribution of a strictly stationary and weakly dependent random process. Unlike the test of Bai and Ng (2005), however, which is based on the classical Charlier–Edgeworth skewness coefficient (standardized third central moment), our test utilizes a robust measure of skewness involving a finite number of quantiles of the marginal distribution of the data. Skewness measures based on quantiles can be traced back to Yule (1911), and their use in problems relating to symmetry of i.i.d. data has been considered by several authors, including, for example, David and Johnson (1956), Hinkley (1975) and Doksum *et al.* (1977). Such measures are particularly useful when the underlying distribution is heavy-tailed or there are extreme observations in the sample (e.g. Kim and White, 2004), characteristics that are often present in economic and financial data. By contrast, the classical moment-based measure of skewness is known to be adversely affected by leptokurtosis and outliers (see, e.g. Horsewell and Looney, 1993; Rayner *et al.*, 1995; Kim and White, 2004).

The quantile-based test considered here has several features that make it attractive for applications. First, in addition to having an intuitive interpretation and being relatively easy to carry out, the test statistic has a standard asymptotic null distribution for a large class of weakly dependent processes satisfying mild mixing conditions. Second, the test does not require any moment assumptions on the observed process. By way of comparison, a test based on the conventional skewness coefficient is applicable only if the marginal distribution of the data has a finite sixth moment. This requirement rules out many economic and financial time series (e.g. equity returns, exchange rate returns and interest rates) since it is often argued that unconditional moments of order higher than two may not be finite for such series (see, e.g. Koedijk *et al.*, 1990; de Lima, 1997). Third, the test has very good size and power properties in samples of sizes that are typical in applications. Finally, as noted earlier, the test is based on a measure of skewness that is more robust to the presence of leptokurtosis and outliers than moment-based measures.

The article is organized as follows. Section 2 introduces a robust measure of skewness and the related test for symmetry, and discusses issues related to the selection of the number of quantiles and the application of the test to fitted residuals. Section 3 examines the finite-sample properties of the proposed test by means of Monte Carlo experiments. Section 4 presents an application to real-world time series. Section 5 summarizes and concludes. Proofs of the main results are placed in the Appendix.

2. FORMULATION OF THE PROBLEM AND THE TEST

2.1. A Quantile-based Measure of Skewness

Let $\{Y_t, t \in \mathbb{Z}\}$ be a strictly stationary sequence of random variables with common distribution function $F(y) = \mathbb{P}(Y_1 \leq y)$, $y \in \mathbb{R}$. The problem of interest is to test the hypothesis that F is symmetric about its (unknown) median ζ, that is,

$$F(y) = 1 - F((2\zeta - y)-) \quad \text{for all } \ y \in \mathbb{R}. \tag{1}$$

The test for symmetry considered here relies on a measure of skewness that is based on a finite number of quantiles of F. Letting $\xi_p = F^{-1}(p)$, $p \in (0,1)$, denote the pth quantile of F, it is easy to see that $\xi_{1/2} - \xi_p = \xi_{1-p} - \xi_{1/2}$ when (1) holds (here and throughout, $\psi^{-1}(u) = \inf\{x : \psi(x) \geq u\}$ for any nondecreasing function ψ). Motivated by this observation, we consider the following measure of skewness:

$$S_k = \boldsymbol{\delta}_k' \boldsymbol{\xi}_k, \tag{2}$$

where $\boldsymbol{\xi}_k = (\xi_{p_1}, \ldots, \xi_{p_k}, \xi_{1/2}, \xi_{1-p_1}, \ldots, \xi_{1-p_k})'$ for some fixed integer $k \geq 1$ and constants $0 < p_1 < \cdots < p_k < \frac{1}{2}$, and $\boldsymbol{\delta}_k$ is a $(2k+1) \times 1$ fixed selection vector such that $S_k = 0$ when (1) holds. One possible choice for $\boldsymbol{\delta}_k$, suggested by Hinkley (1975) and used in Sections 3 and 4, is to put $\boldsymbol{\delta}_{k,k+1} = -2$ and $\boldsymbol{\delta}_{k,i} = 1/k$ for $i \neq k+1$ (here and elsewhere, $\mathbf{v}_{s,i}$ indicates the ith component of a vector $\mathbf{v}_s$).

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Measures of skewness related to S_k were considered by Hinkley (1975) within the context of Box–Cox transformations of i.i.d. data designed to induce approximate symmetry. The skewness test of David and Johnson (1956) is also based on a scaled version of S_k with $k = 1$, $p_1 = 0.0125$ and $\boldsymbol{\delta}_k = (1, -2, 1)'$. Kim and White (2004) demonstrated, by means of Monte Carlo experiments and empirical examples, that quantile-based measures of skewness are not as sensitive to outliers as conventional moment-based measures; they did not, however, consider formal tests based on variants of (2).

Given observations $(Y_1, Y_2, \ldots, Y_T)$, a natural estimator of ξ_p is the empirical pth quantile $\hat{\xi}_p = \hat{F}^{-1}(p)$, where $\hat{F}(y) = T^{-1} \sum_{t=1}^{T} \mathbb{I}(Y_t \leq y)$, $y \in \mathbb{R}$, is the empirical distribution function and $\mathbb{I}(E)$ denotes the indicator of the event E. Such an estimator is (strongly) consistent and asymptotically normal under mild conditions allowing for weak dependence.

In what follows, it will be assumed that, in addition to being strictly stationary, $\{Y_t\}$ is also α-mixing (or strong-mixing). This is the weakest of the classical mixing conditions and is satisfied by a wide variety of linear and nonlinear processes (Doukhan, 1994). Putting $\mathcal{P}_k = \{p_1, \ldots, p_k, 1 - p_1, \ldots, 1 - p_k, \frac{1}{2}\}$ and letting $\{\alpha_n, n \geq 1\}$ denote the α-mixing coefficients of $\{Y_t\}$, we have the following result for $\hat{\boldsymbol{\xi}}_k = (\hat{\xi}_{p_1}, \ldots, \hat{\xi}_{p_k}, \hat{\xi}_{1/2}, \hat{\xi}_{1-p_1}, \ldots, \hat{\xi}_{1-p_k})'$.

Theorem 1. Suppose that $\alpha_n \to 0$ as $n \to \infty$ and $F(\xi_p - \epsilon) < p < F(\xi_p + \epsilon)$ for each $p \in \mathcal{P}_k$ and every $\epsilon > 0$. Then, $\hat{\boldsymbol{\xi}}_k \stackrel{a.s.}{\longrightarrow} \boldsymbol{\xi}_k$ as $T \to \infty$.

It is worth noting that the conclusion of Theorem 1 remains true if α-mixing is replaced by any other dependence condition under which, for each $y \in \mathbb{R}$, $\hat{F}(y) \stackrel{a.s.}{\longrightarrow} F(y)$ as $T \to \infty$.

By strengthening the requirement on the mixing coefficients and imposing some smoothness on F, joint asymptotic normality of the components of $\hat{\boldsymbol{\xi}}_k$ can also be established.

Theorem 2. Suppose that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and that, for each $p \in \mathcal{P}_k$, F is differentiable at ξ_p with $f(\xi_p) = F'(\xi_p) > 0$. Then, $\sqrt{T}(\hat{\boldsymbol{\xi}}_k - \boldsymbol{\xi}_k) \stackrel{d}{\to} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_k)$ as $T \to \infty$, where $\boldsymbol{\Sigma}_k = (\sigma_{ij})$ with

$$\sigma_{ij} = \frac{1}{f(\boldsymbol{\xi}_{k,i}) f(\boldsymbol{\xi}_{k,j})} \left\{ \gamma_{ij}(0) + \sum_{h=1}^{\infty} [\gamma_{ij}(h) + \gamma_{ji}(h)] \right\}, \quad 1 \leq i, j \leq 2k + 1,$$

and $\gamma_{ij}(h) = \mathrm{Cov}[\mathbb{I}(Y_1 \leq \boldsymbol{\xi}_{k,i}), \mathbb{I}(Y_{1+h} \leq \boldsymbol{\xi}_{k,j})]$, $h \geq 0$.

The differentiability condition on F in Theorem 2 is not overly restrictive and is a standard requirement in the literature, for asymptotic normality does not hold unless F is differentiably smooth at the quantiles of interest (cf. Lahiri, 1992; Shapirov and Wendler, 2013).[1] The summability requirement for the mixing coefficients is the best available condition for the central limit theorem for α-mixing processes.

2.2. A Test for Symmetry

Since $S_k = 0$ when F is symmetric, our statistic for testing the null hypothesis of marginal symmetry of $\{Y_t\}$ is defined as

$$\mathrm{QS}_k = \frac{T(\boldsymbol{\delta}_k' \hat{\boldsymbol{\xi}}_k)^2}{\boldsymbol{\delta}_k' \hat{\boldsymbol{\Sigma}}_k \boldsymbol{\delta}_k}, \tag{3}$$

[1] It is interesting to note that, as in the i.i.d. case, the rate of normal approximation to the distribution of empirical quantiles is $O(T^{-1/2})$ under a suitable polynomial α-mixing rate (Lahiri and Sun, 2009).

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where $\hat{\boldsymbol{\Sigma}}_k$ is a consistent estimator of $\boldsymbol{\Sigma}_k$. The following result may be easily established using Theorem 2 and standard arguments.

Theorem 3. Suppose that the conditions of Theorem 2 hold, $\boldsymbol{\Sigma}_k$ is nonsingular and F satisfies (1). Then, $\mathrm{QS}_k \xrightarrow{d} \chi_1^2$ as $T \to \infty$.

In implementing the proposed test based on the statistic in (3), a consistent estimator of $\boldsymbol{\Sigma}_k$ is required. Motivated by the literature on estimation of asymptotic covariance matrices in the presence of weak dependence, we consider an estimator $\hat{\boldsymbol{\Sigma}}_k = (\hat{\sigma}_{ij})$ with

$$\hat{\sigma}_{ij} = \frac{1}{\hat{f}(\hat{\boldsymbol{\xi}}_{k,i})\hat{f}(\hat{\boldsymbol{\xi}}_{k,j})}\left\{\hat{\gamma}_{ij}(0) + \sum_{h=1}^{T-1} W(h/m)\left[\hat{\gamma}_{ij}(h) + \hat{\gamma}_{ji}(h)\right]\right\}, \quad 1 \le i, j \le 2k+1, \tag{4}$$

where

$$\hat{\gamma}_{ij}(h) = T^{-1}\sum_{t=1}^{T-h}\{\mathbb{I}(Y_t \le \hat{\boldsymbol{\xi}}_{k,i}) - \bar{Y}_i^*\}\{\mathbb{I}(Y_{t+h} \le \hat{\boldsymbol{\xi}}_{k,j}) - \bar{Y}_j^*\}, \qquad 0 \le h \le T-1,$$

$$\bar{Y}_i^* = T^{-1}\sum_{t=1}^{T}\mathbb{I}(Y_t \le \hat{\boldsymbol{\xi}}_{k,i}), \quad 1 \le i \le 2k+1,$$

W is a kernel bounded by 1, m is a real-valued bandwidth such that $m \to \infty$ and $m/T \to 0$ as $T \to \infty$, and $\hat{f}$ is a consistent estimator of $f = F'$. Assuming that f is a density for F, it is estimated by means of the Parzen–Rosenblatt estimator

$$\hat{f}(y) = \frac{1}{Tb}\sum_{t=1}^{T} K\left(\frac{y - Y_t}{b}\right), \qquad y \in \mathbb{R}, \tag{5}$$

where K is a nonnegative kernel and b is a positive bandwidth such that $b \to 0$ and $Tb \to \infty$ as $T \to \infty$.

In view of Theorems 1 and 2, consistency of the estimator in (4) follows from well-known results on covariance matrix estimation (see Andrews, 1991; Hansen, 1992; and de Jong, 2000, *inter alia*), combined with uniform consistency of the kernel density estimator in (5). Since $\{\mathbb{I}(Y_t \le \xi_p)\}$ are uniformly bounded, $\alpha_n = O(n^{-\beta})$ for some $\beta > r/2$ and $2 < r \le 4$, coupled with $m = o(T^{1/2-1/r})$, is sufficient for $\hat{\gamma}_{ij}(0) + \sum_{h=1}^{T-1} W(h/m)[\hat{\gamma}_{ij}(h) + \hat{\gamma}_{ji}(h)]$ to converge in probability to $\gamma_{ij}(0) + \sum_{h=1}^{\infty}[\gamma_{ij}(h) + \gamma_{ji}(h)]$ for a large class of absolutely integrable kernels W (cf. de Jong, 2000, Thm 2). Regularity conditions that ensure uniform consistency of $\hat{f}$ can be found in the works of Cai and Roussas (1992), Liebscher (1996) and Hansen (2008), among others. A mixing rate $\alpha_n = O(n^{-\rho})$ with $\rho > 3$ and some smoothness conditions on f are typically sufficient for such results to hold. A set of regularity conditions under which $\hat{f}(\hat{\xi}_p)$ converges almost surely to $f(\xi_p)$, for each $p \in \mathcal{P}_k$, is given in the Appendix.

Needless to say, there are many choices available for suitable kernels W and K that may be used in (4) and (5). The differences in the resulting estimators are not generally substantial (see, e.g. Andrews, 1991; Silverman, 1986, Sec. 3.3.2), and we shall use the Bartlett kernel $W(x) = \max\{0, 1-|x|\}$ and the Gaussian kernel $K(x) = (2\pi)^{-1/2}e^{-x^2/2}$ in the sequel. Regarding the parameters m and b, the former will be selected by means of the data-dependent method of Newey and West (1994); for the latter, the popular Gaussian reference bandwidth $b = 0.79(\hat{\xi}_{3/4} - \hat{\xi}_{1/4})T^{-1/5}$ will be used (cf. Silverman, 1986, Sec. 3.4.2).[2]

[2] Of course, many other methods for selecting b are available in the literature (see, e.g. Jones *et al.*, 1996). We found in our simulations that the finite-sample properties of the test based on QS_k are fairly robust with respect to different data-dependent bandwidth selection methods,

J. Time. Ser. Anal. (2015)
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Finally, we note that, instead of covariance estimators of the type given in (4), a bootstrap estimator of the asymptotic covariance matrix Σ_k may be used. Sun and Lahiri (2006) showed that, under a suitable α-mixing condition and mild smoothness conditions on F, the asymptotic variance of an empirical quantile can be consistently estimated by a blockwise bootstrap method. The blockwise bootstrap may also be used to estimate consistently the distribution of an empirical quantile (Sun and Lahiri, 2006; Shapirov and Wendler, 2013). Although such techniques could be adapted to the problem of testing for symmetry using the statistic in (3), bootstrap-based versions of our test will not be investigated here.

2.3. Determining the Number of Quantiles

An important practical consideration for the application of the test based on QS_k is the selection of the number k of considered empirical quantiles. Obvious possibilities are to fix k at a value independent of T , choose k as a deterministic function of T or compute the value of QS_k for several values of k. However, the first two rules are somewhat arbitrary, while the third option may lead to conflicting evidence being obtained among different values of k.

As a way of avoiding these difficulties, we propose the use of a data-based selection procedure. Restricting attention to evenly spaced quantiles, our suggestion is to determine k by minimizing the estimated asymptotic variance of the empirical skewness measure $\boldsymbol{\delta}'_k \hat{\boldsymbol{\xi}}_k$ adjusted by a penalty term that is an increasing function of k. More precisely, k is chosen so as to minimize an objective function of the form

$$\phi(k) = \ln(\boldsymbol{\delta}'_k \hat{\boldsymbol{\Sigma}}_k \boldsymbol{\delta}_k) + kC_T, \quad 1 \le k \le \bar{k}, \tag{6}$$

where $C_T > 0$ is a sequence of constants that determine the strength of the penalty associated with any given value of k and $\bar{k}$ is a prespecified upper bound for k. The function $\phi(k)$ is reminiscent of various information-based selection criteria that are used widely in a variety of settings. By analogy with such criteria, possible choices for the penalty factor C_T in (6) are $C_T = 2T^{-1}$ and $C_T = T^{-1} \ln T$, which are the values associated with the Akaike information criterion (Akaike, 1974) and the Bayesian information criterion (Rissanen, 1978; Schwarz, 1978) respectively. Akaike's factor penalizes the use of additional quantiles less stringently and, as a result, is likely to lead to larger values for the selected k. In the implementation of this procedure in Sections 3 and 4 , we shall set $\bar{k}$ equal to the integer part of $\sqrt{T}$ and require that the selected quantiles be evenly spaced over the range $0.05 \le p_1 < \cdots < p_k < 0.5$.[3]

2.4. Testing Residuals

Here, we discuss briefly the application of a quantile-based test for symmetry to estimated residuals $\{\hat{\varepsilon}_t, 1 \le t \le T\}$ from a model with strictly stationary, zero mean errors $\{\varepsilon_t\}$ having common distribution function F_ε . We put $\hat{F}_{\hat{\varepsilon}}(x) = T^{-1} \sum_{t=1}^{T} \mathbb{I}(\hat{\varepsilon}_t \le x)$ and $\hat{F}_{\varepsilon}(x) = T^{-1} \sum_{t=1}^{T} \mathbb{I}(\varepsilon_t \le x)$, $x \in \mathbb{R}$, for the empirical distribution functions of the residuals and the errors respectively, and define the associated empirical processes as $V_{\hat{\varepsilon}}(x) = \sqrt{T}\{\hat{F}_{\hat{\varepsilon}}(x) - F_\varepsilon(x)\}$ and $V_\varepsilon(x) = \sqrt{T}\{\hat{F}_\varepsilon(x) - F_\varepsilon(x)\}$.

Under general conditions, the asymptotic distribution of the quantiles of $\hat{F}_{\hat{\varepsilon}}$ may be deduced from the weak limit, as $T \to \infty$, of $V_{\hat{\varepsilon}}$ using the Hadamard differentiability of the quantile map and the functional delta method (see, e.g. van der Vaart and Wellner, 1996, Ch. 3.9). If $V_{\hat{\varepsilon}}$ converges weakly (in a suitable metric space) to a Gaussian process, then residual quantiles will be asymptotically normal. The convergence properties of $V_{\hat{\varepsilon}}$ have been obtained for a variety of parametric models, mostly under an i.i.d. assumption about $\{\varepsilon_t\}$ (e.g. Koul, 2002).

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and so we focus here on the computationally simple Gaussian reference bandwidth selector. It is also worth noting that bandwidth selectors designed for i.i.d. data often work equally well under dependence (Hall *et al.*, 1995).

[3] The restriction on the range of quantiles is imposed because non-Gaussian weak limits are to be expected for extreme empirical quantiles $\hat{\xi}_p$ with $p \to 0$ or $p \to 1$ (Beirlant *et al.*, 2004).

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Table I. Parameters of generalized lambda distribution

	λ_1	λ_2	λ_3	λ_4	Skewness	Kurtosis
S1	0.000000	−1.000000	−0.080000	−0.080000	0.0	6.0
S2	0.000000	−0.397912	−0.160000	−0.160000	0.0	11.6
S3	0.000000	−1.000000	−0.240000	−0.240000	0.0	126.0
A1	0.000000	1.000000	−0.007500	−0.030000	−1.5	7.5
A2	0.000000	1.000000	−0.100900	−0.180200	−2.0	21.1
A3	0.000000	1.000000	−0.001000	−0.130000	−3.16	23.8

The cost of using residuals is that $V_{\hat{\varepsilon}}$ and V_{ε} do not generally converge to the same limit process.[4] There are, however, models in which $V_{\hat{\varepsilon}}$ behaves asymptotically the same as V_{ε}. Examples include autoregressive models (Boldin, 1983), general finite-parameter models for invertible linear processes (Kreiss, 1991), autoregressive moving average models (Bai, 1994), regression models satisfying certain conditions (Lee and Wei, 1999) and autoregressive models with time-varying parameters (Chandler and Polonik, 2012). In such cases, under appropriate regularity conditions and i.i.d. errors, one obtains the well-known weak convergence of the quantile process $\sqrt{T}(\hat{F}_{\varepsilon}^{-1} - F_{\varepsilon}^{-1})$ and hence of $\sqrt{T}(\hat{F}_{\hat{\varepsilon}}^{-1} - F_{\varepsilon}^{-1})$ to the Gaussian process $G/(F'_{\varepsilon} \circ F_{\varepsilon}^{-1})$, where G is a Brownian bridge on [0, 1]. As a result, the asymptotic distribution of a vector of a finite number of quantiles of $\hat{F}_{\hat{\varepsilon}}$ coincides with the Gaussian asymptotic distribution of the corresponding quantiles of $\hat{F}_{\varepsilon}$, and a quantile-based symmetry test is not affected by using $\{\hat{\varepsilon}_t\}$ in lieu of $\{\varepsilon_t\}$.

3. MONTE CARLO SIMULATIONS

The size and power properties of the proposed quantile-based symmetry test are assessed by means of Monte Carlo experiments. The latter are based on artificial data generated according to the following models:

M1: $Y_t = \varepsilon_t$
M2: $Y_t = 0.5Y_{t-1} + \varepsilon_t$
M3: $Y_t = 0.8Y_{t-1} - 0.5\varepsilon_{t-1} + \varepsilon_t$
M4: $Y_t = 1 + 0.5Y_{t-1} + \eta_t\varepsilon_t,\ \eta_t^2 = 0.4 + (0.1\varepsilon_{t-1}^2 + 0.5)\eta_{t-1}^2$

In each case, $\{\varepsilon_t\}$ are i.i.d. random variables having zero mean and unit variance (if the latter exists). Models M2 and M3 are linear, while M4 allows for autoregressive conditional heteroskedasticity. Asymmetry of the distribution of ε_t implies asymmetry of the marginal distribution of $\{Y_t\}$ in all models. The distribution of ε_t is either Gaussian or belongs to the family of generalized lambda distributions. The latter may be specified via its quantile function, which is $F_{\varepsilon}^{-1}(p) = \lambda_1 + \{p^{\lambda_3} - (1-p)^{\lambda_4}\}/\lambda_2$ (Ramberg and Schmeiser, 1974); the parameter values used in the experiments are taken from Bai and Ng (2005) and can be found in Table I. We also consider models with errors that have a stable distribution with characteristic exponent 1.5, location parameter 0, scale parameter 1 and skewness parameter either 0 or −0.8 (Chambers *et al.*, 1976); in this case, $\mathbb{E}(|\varepsilon_t|^{\kappa}) < \infty$ only for $\kappa < 1.5$. Designs with symmetric and asymmetric stable errors are denoted by S4 and A4 respectively, in the following tables.

Experiments proceed by generating 1000 independent artificial time series $\{Y_t\}$ of length $100 + T$, with $T \in \{200, 500\}$, for each design point. The first 100 data points of each series are then discarded in order to eliminate start-up effects, while the remaining T data points are used to compute the value of the QS_k statistic defined in (3). For the latter, k is selected by means of the data-based procedure described in Section 2.3, setting $C_T = 2T^{-1}$ or $C_T = T^{-1}\ln T$ in (6); the corresponding tests are labelled $\mathrm{QS}_k(A)$ and $\mathrm{QS}_k(B)$.

[4] Ghoudi and Remillard (1998) provide some general results on empirical processes based on 'pseudo-observations'.

J. Time. Ser. Anal. (2015)
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The quantile-based test is compared with the test discussed by Bai and Ng (2005), which is based on the statistic

$$\mathrm{BN} = \frac{T\hat{\mu}_3^2}{\hat{\mu}_2^3\hat{\omega}^2},$$

where $\hat{\mu}_s = T^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})^s$ for $s \geq 2$, $\bar{Y} = T^{-1}\sum_{t=1}^{T} Y_t$, and $\hat{\omega}^2$ is an estimator of the asymptotic variance of $\sqrt{T}\hat{\mu}_3\hat{\mu}_2^{-3/2}$ that is consistent under symmetry (as in the case of QS_k, we use a Bartlett kernel estimator with a data-dependent bandwidth). Under appropriate mixing and moment conditions (including $\mathbb{E}(|Y_1|^{\kappa}) < \infty$ for some $\kappa > 6$), $\mathrm{BN} \xrightarrow{d} \chi_1^2$ as $T \to \infty$ under (1).

The Monte Carlo rejection frequencies of tests of nominal level 0.05 are reported in Table II (the number of empirical quantiles used to construct $\mathrm{QS}_k(A)$ and $\mathrm{QS}_k(B)$, averaged across replications, is also included). The results show the following:

(i) For most design points, the tests based on QS_k have empirical levels that do not differ significantly from the nominal level. A small level distortion is observed in a few cases involving leptokurtic symmetric errors, but even in such cases, the distortion is not of a magnitude that makes the tests unattractive.

(ii) The two QS_k tests have very good power properties, even for the smallest of the sample sizes considered. As reported in other studies involving different tests for symmetry, leptokurtosis in the errors appears to

Table II. Empirical rejection frequencies of tests for symmetry

		$T=200$					$T=500$				
	Distr.	BN	$\mathrm{QS}_k(A)$	k	$\mathrm{QS}_k(B)$	k	BN	$\mathrm{QS}_k(A)$	k	$\mathrm{QS}_k(B)$	k
M1	N	0.06	0.06	4.5	0.04	3.6	0.04	0.05	5.2	0.05	4.2
	S1	0.03	0.07	6.6	0.06	5.0	0.05	0.05	8.5	0.06	6.5
	S2	0.03	0.05	7.6	0.07	5.8	0.04	0.07	10.3	0.05	7.4
	S3	0.04	0.06	8.6	0.07	6.5	0.03	0.07	12.2	0.07	8.5
	S4	0.01	0.05	9.5	0.06	7.2	0.01	0.06	14.2	0.06	9.6
	A1	0.90	0.94	6.5	0.94	5.2	0.98	1.00	8.5	1.00	6.5
	A2	0.47	0.62	7.5	0.63	5.9	0.73	0.95	10.5	0.95	7.5
	A3	0.73	1.00	7.9	1.00	6.4	0.88	1.00	11.4	1.00	8.2
	A4	0.19	0.77	9.0	0.79	7.2	0.18	0.99	13.8	0.99	9.8
M2	N	0.06	0.06	4.8	0.05	3.8	0.06	0.05	5.7	0.05	4.5
	S1	0.05	0.07	5.9	0.05	4.5	0.06	0.05	7.9	0.04	5.8
	S2	0.04	0.05	6.6	0.07	5.1	0.04	0.05	9.4	0.07	6.8
	S3	0.05	0.06	7.6	0.08	5.7	0.03	0.07	10.8	0.06	7.6
	S4	0.02	0.06	10.3	0.07	8.1	0.01	0.07	16.4	0.06	11.4
	A1	0.84	0.71	6.5	0.73	4.9	0.97	0.96	8.3	0.98	6.2
	A2	0.44	0.40	6.9	0.42	5.4	0.73	0.76	9.7	0.80	7.1
	A3	0.76	0.98	7.9	0.99	6.1	0.87	1.00	11.6	1.00	8.2
	A4	0.20	0.60	9.7	0.63	7.7	0.21	0.94	15.2	0.96	11.4
M3	N	0.07	0.06	4.8	0.05	3.8	0.07	0.05	5.9	0.05	4.5
	S1	0.04	0.05	5.8	0.07	4.6	0.05	0.05	7.8	0.06	5.7
	S2	0.05	0.06	6.5	0.07	5.0	0.04	0.06	8.8	0.07	6.4
	S3	0.02	0.06	7.0	0.08	5.4	0.04	0.07	10.3	0.06	7.4
	S4	0.02	0.06	9.4	0.07	7.8	0.01	0.06	16.1	0.06	11.8
	A1	0.85	0.71	6.1	0.74	4.7	0.96	0.96	8.3	0.96	5.9
	A2	0.46	0.37	6.7	0.43	5.1	0.71	0.73	9.2	0.75	6.7
	A3	0.74	0.99	7.6	0.98	5.9	0.87	1.00	10.4	1.00	7.8
	A4	0.18	0.56	9.0	0.55	7.3	0.19	0.91	14.5	0.91	11.2
M4	N	0.07	0.06	5.1	0.06	4.0	0.05	0.05	6.5	0.06	5.0
	S1	0.06	0.07	6.5	0.07	5.1	0.04	0.06	9.1	0.06	6.6
	S2	0.04	0.08	7.1	0.05	5.6	0.03	0.06	10.6	0.05	7.5
	S3	0.05	0.06	8.4	0.07	6.2	0.03	0.05	12.2	0.07	8.5
	S4	0.02	0.06	10.3	0.07	8.2	0.02	0.06	16.4	0.06	11.8
	A1	0.78	0.72	6.6	0.74	5.2	0.90	0.98	9.9	0.99	7.1
	A2	0.38	0.43	7.6	0.40	5.8	0.65	0.75	10.9	0.79	7.9
	A3	0.71	0.99	8.2	0.98	6.6	0.81	1.00	12.8	1.00	9.1
	A4	0.19	0.61	9.7	0.61	8.0	0.17	0.95	15.5	0.96	11.6

Table III. Tests for marginal symmetry

Company	BN	$QS_k(A)$	k	$QS_k(B)$	k	Company	BN	$QS_k(A)$	k	$QS_k(B)$	k
Alcoa Inc	0.12	0.61	8	0.61	8	Danaher Corp.	0.26	0.01	11	0.01	7
Apple Inc.	0.75	0.94	9	0.98	5	Walt Disney Co.	0.89	0.08	10	0.08	10
Adobe Systems Inc	0.36	0.14	18	0.14	9	Dow Chemical	0.39	0.15	7	0.22	5
Analog Devices Inc	0.19	0.64	10	0.64	10	Duke Energy	0.37	0.11	7	0.11	4
Archer-Daniels-Midland Co	0.06	0.20	6	0.20	6	Ecolab Inc.	0.56	0.00	4	0.00	4
Autodesk Inc	0.10	0.90	7	0.90	7	Equifax Inc.	0.43	0.05	16	0.02	6
American Electric Power	0.44	0.47	19	0.47	11	Edison Int'l	0.87	0.09	18	0.09	13
AES Corp	0.30	0.89	22	0.86	12	EMC Corp.	0.35	0.51	11	0.60	9
AFLAC Inc	0.02	0.00	13	0.00	13	Emerson Electric	0.39	0.55	6	0.55	6
Allergan Inc	0.80	0.03	15	0.03	6	Equity Residential	0.03	0.14	10	0.14	10
American Intl Group Inc	0.14	0.20	9	0.20	9	EQT Corporation	0.12	0.12	9	0.10	8
Aon plc	0.28	0.10	15	0.07	8	Eaton Corp.	0.34	0.99	6	0.99	6
Apache Corporation	0.36	0.26	8	0.26	8	Entergy Corp.	0.81	0.11	6	0.11	6
Anadarko Petroleum Corp	0.18	0.49	7	0.49	7	Exelon Corp.	0.12	0.29	9	0.25	6
Avon Products	0.50	0.30	7	0.30	7	Ford Motor	0.68	0.07	6	0.11	4
Avery Dennison Corp	0.74	0.25	7	0.25	7	Fastenal Co	0.98	0.97	5	0.97	5
American Express Co	0.72	0.09	16	0.12	10	Family Dollar Stores	0.43	0.08	5	0.08	5
Bank of America Corp	0.60	0.18	16	0.20	9	FedEx Corporation	0.18	0.26	10	0.31	8
Baxter International Inc.	0.25	0.10	4	0.10	4	Fiserv Inc	0.20	0.07	5	0.07	5
BBT Corporation	0.08	0.97	13	0.87	8	Fifth Third Bancorp	0.15	0.06	10	0.06	10
Best Buy Co. Inc.	0.29	0.00	9	0.01	5	Fluor Corp.	0.29	0.06	8	0.06	8
Bard (C.R.) Inc.	0.37	0.60	14	0.60	12	Forest Laboratories	0.33	0.96	9	0.93	7
Becton Dickinson	0.83	0.89	18	0.87	10	Frontier Communications	0.79	0.28	16	0.26	11
Franklin Resources	0.37	0.62	14	0.58	7	Gannett Co.	0.07	0.06	10	0.06	10
Brown-Forman Corporation	0.05	0.17	8	0.08	4	General Dynamics	0.85	0.09	11	0.09	6
Baker Hughes Inc	0.93	0.63	5	0.63	5	General Electric	0.38	0.24	7	0.24	7
The Bank of NY Mellon Corp.	0.43	0.29	5	0.29	5	General Mills	0.71	0.64	17	0.61	11
Ball Corp	0.75	0.08	13	0.06	7	Genuine Parts	0.08	0.16	7	0.16	7
Boston Scientific	0.61	0.57	12	0.53	9	Gap (The)	0.38	0.52	16	0.37	8
Cardinal Health Inc.	0.54	0.92	11	0.98	8	Grainger Inc.	0.27	0.83	12	0.78	6
Caterpillar Inc.	0.21	0.43	7	0.29	4	Halliburton Co.	0.92	0.46	11	0.53	8
Chubb Corp.	0.04	0.48	7	0.48	7	Harman Int'l Industries	0.20	0.00	20	0.00	11
Coca-Cola Enterprises	0.93	0.08	14	0.07	10	Hasbro Inc.	0.72	0.11	17	0.13	12
Carnival Corp.	0.43	0.04	7	0.04	7	Huntington Bancshares	0.33	0.91	6	0.91	6
CIGNA Corp.	0.38	0.67	10	0.67	10	Health Care REIT	0.83	0.01	8	0.01	5
Cincinnati Financial	0.12	0.75	15	0.77	8	Home Depot	0.47	0.09	7	0.09	7
Clorox Co.	0.11	0.07	24	0.08	10	Hess Corporation	0.50	0.37	7	0.37	7
Comerica Inc.	0.92	0.67	11	0.68	8	Harley-Davidson	0.11	0.02	11	0.02	6
CMS Energy	0.39	0.96	8	0.96	8	Honeywell Int'l Inc.	0.45	0.40	15	0.42	8
CenterPoint Energy	0.88	0.09	11	0.09	11	Hewlett-Packard	0.89	0.54	9	0.40	5
Cabot Oil and Gas	0.04	0.06	12	0.06	12	Block H and R	0.36	0.39	5	0.39	5
ConocoPhillips	0.77	0.56	6	0.70	4	Hormel Foods Corp.	0.97	0.46	16	0.51	7
Campbell Soup	0.42	0.70	10	0.71	7	The Hershey Company	0.27	0.50	6	0.67	5
CSX Corp.	0.90	0.85	10	0.85	10	Intel Corp.	0.21	0.13	11	0.13	7
CenturyLink Inc	0.73	0.26	23	0.22	11	International Paper	0.50	0.03	14	0.04	7
Cablevision Systems Corp.	0.09	0.09	18	0.08	13	Interpublic Group	0.09	0.85	13	0.82	10
Chevron Corp.	0.04	0.81	4	0.81	4	Ingersoll-Rand PLC	0.67	0.91	8	0.91	8
Dominion Resources	0.18	0.19	12	0.24	9	Johnson Controls	0.86	0.36	12	0.28	7
Deere and Co.	0.50	0.78	8	0.85	5	Jacobs Engineering Group	0.98	0.56	14	0.63	8
D. R. Horton	0.01	0.01	9	0.01	5	Johnson and Johnson	0.04	0.28	11	0.37	5

J. Time. Ser. Anal. (2015)
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have a deleterious effect on power. For example, in the case of M2, the rejection frequency of $\mathrm{QS}_k(B)$ is 0.73 under A1 when $T = 200$, but drops to 0.42 under A2 ($\mathrm{QS}_k(A)$ and BN exhibit similar behaviour). However, it is worth pointing out that the power of the QS_k tests quickly improves as the sample size increases; as a matter of fact, power is always in excess of 0.73 when $T = 500$.

(iii) As expected, the average number of quantiles selected by the data-dependent rule is higher when an Akaike-type penalty is used to construct $\phi(k)$; it ranges from 4 to 17 in the case of $\mathrm{QS}_k(A)$ and from 3 to 12 for $\mathrm{QS}_k(B)$, depending on the design point, and is generally higher under non-Gaussianity. Importantly, however, the rejection frequencies of $\mathrm{QS}_k(A)$ and $\mathrm{QS}_k(B)$ are extremely similar, suggesting that the size and power properties of the quantile-based test are insensitive with respect to the penalty factor used in (6).

(iv) Perhaps unsurprisingly, the test based on BN exhibits non-negligible level distortion in the case of stable innovations with infinite variance, which also results in extremely low rejection rates in the cases involving asymmetric stable innovations. For most of the remaining design points, the BN test has lower power than the QS_k tests.

4. EMPIRICAL APPLICATION

The $\mathrm{QS}_k(A)$, $\mathrm{QS}_k(B)$ and BN tests are applied to a set of weekly stock returns, spanning the period 1993–2007, for 100 companies from the Standard & Poor's 500 Composite Index. The selected series are part of the data set analyzed by Kapetanios (2009) and are such that the null hypothesis of strict stationarity cannot be rejected for any of them (at significance level 0.05).

The asymptotic p-values for the tests are reported in Table III. At significance level 0.10 (0.05), evidence against symmetry is found in 31 (12) and 15 (8) series on the basis of the QS_k and BN tests respectively. There are a few series for which only the BN test rejects symmetry. These rejections seem to be related, to some extent, to the presence of extreme outliers in the data; once such outliers are removed, the evidence in favour of asymmetry provided by BN is considerably weaker.[5]

Findings of asymmetry have important implications for portfolio and risk management. For example, according to the Basel banking regulations, commercial banks are required to measure the market risk of their asset portfolios and hold capital in proportion to their risk position. Banks constructing portfolios from significantly negatively skewed asset returns can be systematically exposed to higher downside risk and are required, therefore, to hold more (cash) reserves, something that may lead to reduced overall profitability.

5. SUMMARY

This article considered a quantile-based test for symmetry of the marginal distribution of a strictly stationary and weakly dependent random process. The test is intuitive, easy to implement, requires no moment assumptions on the observed process and has a standard asymptotic null distribution under mild mixing conditions. The Monte Carlo results revealed that the quantile-based test has good size and power properties in finite samples, and outperforms the popular moment-based test for symmetry, especially in the presence of heavy tails. An application to time series of stock returns illustrated the practical use of the test.

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[5] Recalling that the asymptotic validity of the BN test requires finiteness of the sixth moment of F, it is also worth noting that an estimate of the maximal moment exponent $\sup\{\kappa > 0 : \int_{\mathbb{R}} |x|^{\kappa}\, dF(x) < \infty\}$, obtained using the conventional Hill estimator based on the $T/10$ largest empirical quantiles, is less than 6 for 82 out of the 100 series under consideration.

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J. Time. Ser. Anal. (2015)
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APPENDIX

A1. Proof of Theorem 1

Noting that $\{Y_t\}$ is ergodic when $\alpha_n \to 0$ as $n \to \infty$ (e.g. Doukhan, 1994, p. 21), we have $\hat{F}(y) \xrightarrow{a.s.} F(y)$ as $T \to \infty$, uniformly in $y \in \mathbb{R}$, on account of the Glivenko–Cantelli theorem for stationary and ergodic processes (Dehling and Philipp, 2002, Thm 1.1). Hence, by arguing as in the proof of Theorem 2.3.1 of Serfling (1980, p. 75), it can be deduced that $\hat{\xi}_p \xrightarrow{a.s.} \xi_p$ as $T \to \infty$ for each $p \in \mathcal{P}_k$, and the assertion of the theorem follows.

A2. Proof of Theorem 2

By Theorem 1 of Shapirov and Wendler (2013), for each $p \in \mathcal{P}_k$, $\hat{\xi}_p$ admits the Bahadur–Ghosh representation

$$\sqrt{T}(\hat{\xi}_p - \xi_p) = \frac{\sqrt{T}\{p - \hat{F}(\xi_p)\}}{f(\xi_p)} + R_p = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\frac{p - \mathbb{I}\,(Y_t \le \xi_p)}{f(\xi_p)} \right) + R_p,$$

with $R_p = o_{\mathbb{P}}(1)$ as $T \to \infty$. Hence, it suffices to show that $T^{-1/2} \sum_{t=1}^{T} z_t \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_k)$ as $T \to \infty$, where

$$z_t = \left(\frac{p_1 - \mathbb{I}\,(Y_t \le \xi_{p_1})}{f(\xi_{p_1})}, \ldots, \frac{(1/2) - \mathbb{I}\,(Y_t \le \xi_{1/2})}{f(\xi_{1/2})}, \ldots, \frac{1 - p_k - \mathbb{I}\,(Y_t \le \xi_{1-p_k})}{f(\xi_{1-p_k})} \right)'.$$

To this end, note that, for each $p \in \mathcal{P}_k$, $\{p - \mathbb{I}\,(Y_t \le \xi_p)\}$ is a strictly stationary sequence of bounded random variables having zero mean (on account of the assumed continuity of F at ξ_p) and the same α-mixing coefficients as $\{Y_t\}$. Therefore, by appealing to a central limit theorem for α-mixing sequences (cf. Ibragimov and Linnik, 1971, Thm 18.5.4, p. 347), we may conclude that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{p-\mathbb{I}\left(Y_t\le\xi_p\right)}{f(\xi_p)}\right)\xrightarrow{d}\mathcal{N}\left(0,\tau_p^2/f^2(\xi_p)\right)\quad\text{as } T\to\infty,$$

where $\tau_p^2=\sum_{h=-\infty}^{\infty}\text{Cov}[\mathbb{I}(Y_1\le\xi_p),\mathbb{I}(Y_{1+h}\le\xi_p)]<\infty$. (The variance of the normalized partial sum earlier approaches zero as $T\to\infty$ if $\tau_p^2=0$, in which case, the limiting distribution is understood to be a unit mass at zero.) By considering arbitrary linear combinations of the components of $T^{-1/2}\sum_{t=1}^{T}\mathbf{z}_t$ and applying the central limit theorem for α-mixing sequences, the required result follows in a similar manner via the Cramér–Wold device.

A3. Convergence of Density Estimates at Random Points

Consider the following regularity conditions:

(C1) $\{Y_t\}$ is strictly stationary and α-mixing with $\alpha_n=O(n^{-\rho})$ for some $\rho>3$.
(C2) F possesses an everywhere-positive density satisfying a uniform Lipschitz condition of order 1.
(C3) K is an everywhere-differentiable, nonnegative function on $\mathbb{R}$ such that $K(x)\to 0$ as $|x|\to\infty$, $\int_{\mathbb{R}}K(x)dx=1$, $\int_{\mathbb{R}}|x|\,K(x)dx<\infty$, and $\int_{\mathbb{R}}|K'(x)|\,dx<\infty$.
(C4) $b\to 0$ and $Tb^2/\ln\ln T\to\infty$ as $T\to\infty$.

Under conditions (C1)–(C4), the estimator defined by (5) satisfies $\hat{f}(y)\xrightarrow{a.s.}f(y)$ as $T\to\infty$, uniformly in $y\in\mathbb{R}$ (Cai and Roussas, 1992, Thm 4.1). Combined with Theorem 1, this implies that $\hat{f}(\hat{\xi}_p)\xrightarrow{a.s.}f(\xi_p)$ as $T\to\infty$ for each $p\in\mathcal{P}_k$, $\xi=\xi_p$ being the unique root of the equation $F(\xi)=p$.

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